Saint Martin’s University

Project Report

MME 566, A2

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1 Project Formulation:

The figure on the right is of the 2-D truss system to be studied. - Elements are numbered using the circled numbers - Nodes are numbered per the bold/blue numbers

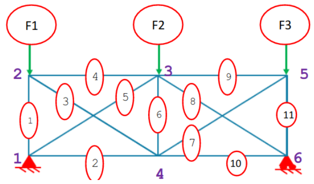

Figure 1: The original truss

1.1 My specific project specifications are as follows:

- Modulus: $E = 100\ MPa$
- Forces: $F_1 = 60N$, $F_2 = 110N$, $F_3 = 60N$
- Element 1:
  - $L_1 = 4m$
  - $A_1 = 4 * 10^{-4}m^2$
- Element 2:
  - $L_2 = 4m$
  - $A_2 = 4 * 10^{-4}m^2$
- Element 3:
  - $L_3 = 5.65m$
  - $A_3 = 4 * 10^{-4}m^2$
- Element 4:
  - $L_4 = 4m$
  - $A_4 = 4 * 10^{-4}m^2$
- Element 5:
  - $L_5 = 5.65m$
  - $A_5 = 4 * 10^{-4}m^2$
- Element 6:
  - $L_6 = 4m$
  - $A_6 = 4 * 10^{-4}m^2$
- Element 7:
  - $L_7 = 5.65m$
  - $A_7 = 4 * 10^{-4}m^2$
- Element 8:
  - $L_8 = 5.65m$
  - $A_8 = 4 * 10^{-4}m^2$
- Element 9:
  - $L_9 = 4m$
  - $A_9 = 4 * 10^{-4}m^2$
- Element 10:
  - $L_{10} = 4m$
  - $A_{10} = 4 * 10^{-4}m^2$

- Element 11:
  - $L_{11} = 4m$
  - $A_{11} = 4 * 10^{-4} m^2$

1.2 The boundary conditions in effect are:

- u_{1x} and u_{1y} are both 0
- u_{6y} is 0
- $\begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \\ f_{3x} \\ f_{3y} \\ f_{4x} \\ f_{4y} \\ f_{5x} \\ f_{5y} \\ f_{6x} \\ f_{6y} \end{Bmatrix} = \begin{Bmatrix} R_{1x} \\ R_{1y} \\ 0 \\ -60N \\ 0 \\ -110N \\ 0 \\ 0 \\ 0 \\ -60N \\ 0 \\ R_{6y} \end{Bmatrix}$

2 Global Stiffness Matrix Derivation

2.0.1 The Generalized Stiffness Matrix

Motivation behind finding a generalized transform: Our forces are nodes are all located in global coordinates, using a reference frame that remains fixed regardless of any motion of the bodies or the orienation of the element. This system can be expressed as:

$$F_G^e = K_G^e U_G^e$$

Where e is the element, K is the stiffness matrix, and F and U are are the force and displacement vectors respectively. They have the subscript 'G' to indicate that they are in the global/inertial reference frame.

The global reference frame has the benefit of being well defined at the surface level, allowing for an intuitive understanding of the forces on the system and simple expression of node displacement.

It is simpler to calculate displacement on a given node (attached to its parent element) if the forces are either aligned to the element(more complicated motion is outside the scope of this analysis). For this reason we would prefer to work in local coordinates. The formula for the local system looks superficially similar to the global system. The local system is distinct in that the composition of its stiffness matrix, k, is known and simple:

$$F_l^e = K_l^e U_l^e$$

where $K = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

This line of thought motivates us towards finding a transformation matrix T, capable of converting from global coordinates to local coordinates. This matrix's inverse would then convert from local to global coordinates.

With such a transformation matrix, our global displacement/force equation looks like so:

$$F_G^e = TF_l^e$$

$$F_G^e = T^{-1}K_l^e TU_g^e$$

Assembling a generalized stiffness matrix The desired transformation from global to local coordinates is expressed as follows:

$$\begin{Bmatrix} u_{1x}^1 \\ u_{1y}^1 \\ u_{2x}^1 \\ u_{2y}^1 \end{Bmatrix} = \begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix} \begin{Bmatrix} u_{1x} \\ u_{1y} \\ u_{2x} \\ u_{2y} \end{Bmatrix} \Rightarrow T^e = \begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix},$$

where $c = cos(\theta)$ and $s = sin(\theta)$

This transformation matrix has the added benefit of converting back from local to global when transposed: its determinant is 1, making its inverse equal to its transpose.

With a transformation matrix defined and knowing that $T^{-1} = T^T$, a global stiffness matrix can be assembled via the following formula:

$$K_G^e = T^T K_l^e T = \begin{bmatrix} c^2 & sc & -c^2 & -sc \\ sc & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -sc & -s^2 & sc & s^2 \end{bmatrix}$$

While potentially being more computationally expensive than formulating a solution for each individual element, this generalized approach is both more robust and faster, once computers are employed.

2.1 The stiffness matrix derivations for individual elements is as follows:

2.1.1 Element 1:

$\theta_1 = \frac{\pi}{2} rad$

$k_1 = \frac{EA_1}{L_1} = 10\frac{kN}{m}$

$$\begin{Bmatrix} f_{1x}^1 \\ f_{1y}^1 \\ f_{2x}^1 \\ f_{2y}^1 \end{Bmatrix} = k_1 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_{1x}^1 \\ u_{1y}^1 \\ u_{2x}^1 \\ u_{2y}^1 \end{Bmatrix}$$

2.1.2 Element 2:

$\theta_2 = 0rad$

$k_2 = \frac{EA_2}{L_2} = 10\frac{kN}{m}$

$$\begin{Bmatrix} f_{1x}^2 \\ f_{1y}^2 \\ f_{4x}^2 \\ f_{4y}^2 \end{Bmatrix} = k_2 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_{1x}^2 \\ u_{1y}^2 \\ u_{4x}^2 \\ u_{4y}^2 \end{Bmatrix}$$

2.1.3 Element 3:

$\theta_3 = -\frac{\pi}{4}rad$

$k_3 = \frac{EA_3}{L_3} = 7.080\frac{kN}{m}$

$$\begin{Bmatrix} f_{2x}^3 \\ f_{2y}^3 \\ f_{4x}^3 \\ f_{4y}^3 \end{Bmatrix} = k_3 \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{Bmatrix} u_{2x}^3 \\ u_{2y}^3 \\ u_{4x}^3 \\ u_{4y}^3 \end{Bmatrix}$$

2.1.4 Element 4:

$\theta_4 = 0rad$

$k_4 = \frac{EA_4}{L_4} = 10\frac{kN}{m}$

$$\begin{Bmatrix} f_{2x}^4 \\ f_{2y}^4 \\ f_{3x}^4 \\ f_{3y}^4 \end{Bmatrix} = k_4 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_{2x}^4 \\ u_{2y}^4 \\ u_{3x}^4 \\ u_{3y}^4 \end{Bmatrix}$$

2.1.5 Element 5:

$\theta_5 = \frac{\pi}{4}rad$

$k_5 = \frac{EA_5}{L_5} = 7.080\frac{kN}{m}$

$$\begin{Bmatrix} f_{1x}^5 \\ f_{1y}^5 \\ f_{3x}^5 \\ f_{3y}^5 \end{Bmatrix} = k_5 \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{Bmatrix} u_{1x}^5 \\ u_{1y}^5 \\ u_{3x}^5 \\ u_{3y}^5 \end{Bmatrix}$$

2.1.6 Element 6:

$\theta_6 = \frac{\pi}{2} rad$

$k_6 = \frac{EA_6}{L_6} = 10 \frac{kN}{m}$

$$\left\{ \begin{array}{c} f_{4x}^6 \\ f_{4y}^6 \\ f_{3x}^6 \\ f_{3y}^6 \end{array} \right\} = k_6 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \left\{ \begin{array}{c} u_{4x}^6 \\ u_{4y}^6 \\ u_{3x}^6 \\ u_{3y}^6 \end{array} \right\}$$

2.1.7 Element 7:

$\theta_7 = \frac{\pi}{4} rad$

$k_7 = \frac{EA_7}{L_7} = 7.080 \frac{kN}{m}$

$$\left\{ \begin{array}{c} f_{4x}^7 \\ f_{4y}^7 \\ f_{5x}^7 \\ f_{5y}^7 \end{array} \right\} = k_7 \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \left\{ \begin{array}{c} u_{4x}^7 \\ u_{4y}^7 \\ u_{5x}^7 \\ u_{5y}^7 \end{array} \right\}$$

2.1.8 Element 8:

$\theta_8 = -\frac{\pi}{4} rad$

$k_8 = \frac{EA_8}{L_8} = 7.080 \frac{kN}{m}$

$$\left\{ \begin{array}{c} f_{3x}^8 \\ f_{3y}^8 \\ f_{6x}^8 \\ f_{6y}^8 \end{array} \right\} = k_8 \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \left\{ \begin{array}{c} u_{3x}^8 \\ u_{3y}^8 \\ u_{6x}^8 \\ u_{6y}^8 \end{array} \right\}$$

2.1.9 Element 9:

$\theta_9 = 0 rad$

$k_9 = \frac{EA_9}{L_9} = 10 \frac{kN}{m}$

$$\left\{ \begin{array}{c} f_{3x}^9 \\ f_{3y}^9 \\ f_{5x}^9 \\ f_{5y}^9 \end{array} \right\} = k_9 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \left\{ \begin{array}{c} u_{3x}^9 \\ u_{3y}^9 \\ u_{5x}^9 \\ u_{5y}^9 \end{array} \right\}$$

2.1.10 Element 10:

$\theta_{10} = 0 rad$

$k_{10} = \frac{EA_{10}}{L_{10}} = 10 \frac{kN}{m}$

$$\begin{Bmatrix} f_{4x}^{10} \\ f_{4y}^{10} \\ f_{6x}^{10} \\ f_{6y}^{10} \end{Bmatrix} = k_{10} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_{4x}^{10} \\ u_{4y}^{10} \\ u_{6x}^{10} \\ u_{6y}^{10} \end{Bmatrix}$$

2.1.11 Element 11:

$\theta_{11} = \frac{\pi}{2} rad$

$k_{11} = \frac{EA_{11}}{L_{11}} = 10 \frac{kN}{m}$

$$\begin{Bmatrix} f_{6x}^{11} \\ f_{6y}^{11} \\ f_{5x}^{11} \\ f_{5y}^{11} \end{Bmatrix} = k_{11} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_{6x}^{11} \\ u_{6y}^{11} \\ u_{5x}^{11} \\ u_{5y}^{11} \end{Bmatrix}$$

2.2 Assembling the Global Stiffness Matrix

$K =$

.

$$\begin{bmatrix}
k_2+\frac{k_5}{2} & \frac{k_5}{2} & 0 & 0 & -\frac{k_5}{2} & -\frac{k_5}{2} & -k_2 & 0 & 0 & 0 & 0 & 0 \\
\frac{k_5}{2} & k_1+\frac{k_5}{2} & 0 & -k_1 & -\frac{k_5}{2} & -\frac{k_5}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{k_3}{2}+k_4 & -\frac{k_3}{2} & -k_4 & 0 & -\frac{k_3}{2} & \frac{k_3}{2} & 0 & 0 & 0 & 0 \\
0 & -k_1 & -\frac{k_3}{2} & k_1+\frac{k_3}{2} & 0 & 0 & \frac{k_3}{2} & -\frac{k_3}{2} & 0 & 0 & 0 & 0 \\
-\frac{k_5}{2} & -\frac{k_5}{2} & -k_4 & 0 & C_1 & \frac{k_5}{2}-\frac{k_8}{2} & 0 & 0 & -k_9 & 0 & -\frac{k_8}{2} & \frac{k_8}{2} \\
-\frac{k_5}{2} & -\frac{k_5}{2} & 0 & 0 & \frac{k_5}{2}-\frac{k_8}{2} & \frac{k_5}{2}+k_6+\frac{k_8}{2} & 0 & -k_6 & 0 & 0 & \frac{k_8}{2} & -\frac{k_8}{2} \\
-k_2 & 0 & -\frac{k_3}{2} & \frac{k_3}{2} & 0 & 0 & C_2 & -\frac{k_3}{2}+\frac{k_7}{2} & -\frac{k_7}{2} & -\frac{k_7}{2} & -k_{10} & 0 \\
0 & 0 & \frac{k_3}{2} & -\frac{k_3}{2} & 0 & -k_6 & -\frac{k_3}{2}+\frac{k_7}{2} & \frac{k_3}{2}+k_6+\frac{k_7}{2} & -\frac{k_7}{2} & -\frac{k_7}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & -k_9 & 0 & -\frac{k_7}{2} & -\frac{k_7}{2} & \frac{k_7}{2}+k_9 & \frac{k_7}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{k_7}{2} & -\frac{k_7}{2} & \frac{k_7}{2} & \frac{k_7}{2}+k_{11} & 0 & -k_{11} \\
0 & 0 & 0 & 0 & -\frac{k_8}{2} & \frac{k_8}{2} & -k_{10} & 0 & 0 & 0 & \frac{k_8}{2}+k_{10} & -\frac{k_8}{2} \\
0 & 0 & 0 & 0 & \frac{k_8}{2} & -\frac{k_8}{2} & 0 & 0 & 0 & -k_{11} & -\frac{k_8}{2} & \frac{k_8}{2}+k_{11}
\end{bmatrix}$$

$C_1 = k_4 + \frac{k_5}{2} + \frac{k_8}{2} + k_9$ and $C_2 = k_2 + \frac{k_3}{2} + \frac{k_7}{2} + k_{10}$

$$\begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \\ f_{3x} \\ f_{3y} \\ f_{4x} \\ f_{4y} \\ f_{5x} \\ f_{5y} \\ f_{6x} \\ f_{6y} \end{Bmatrix} = K \begin{Bmatrix} u_{1x} \\ u_{1y} \\ u_{2x} \\ u_{2y} \\ u_{3x} \\ u_{3y} \\ u_{4x} \\ u_{4y} \\ u_{5x} \\ u_{5y} \\ u_{6x} \\ u_{6y} \end{Bmatrix}$$

3 Applying Boundary Conditions

With boundary conditions applied, the problem becomes:

$$\left\{\begin{matrix} R_{1x} \\ R_{1y} \\ 0 \\ 60N \\ 0 \\ 110N \\ 0 \\ 0 \\ 0 \\ 60N \\ 0 \\ R_{6y} \end{matrix}\right\} = K \left\{\begin{matrix} 0 \\ 0 \\ u_{2x} \\ u_{2y} \\ u_{3x} \\ u_{3y} \\ u_{4x} \\ u_{4y} \\ u_{5x} \\ u_{5y} \\ u_{6x} \\ 0 \end{matrix}\right\}$$

This simplifies the global stiffness matrix, bringing it to:

$$K_{bc} = \begin{bmatrix} \frac{k_3}{2}+k_4 & -\frac{k_3}{2} & -k_4 & 0 & -\frac{k_3}{2} & \frac{k_3}{2} & 0 & 0 & 0 \\ -\frac{k_3}{2} & k_1+\frac{k_3}{2} & 0 & 0 & \frac{k_3}{2} & -\frac{k_3}{2} & 0 & 0 & 0 \\ -k_4 & 0 & C_1 & \frac{k_5}{2}+\frac{k_7}{2}-\frac{k_8}{2} & 0 & 0 & -k_9 & 0 & -\frac{k_8}{2} \\ 0 & 0 & \frac{k_5}{2}-\frac{k_8}{2} & \frac{k_5}{2}+k_6+\frac{k_8}{2} & 0 & -k_6 & 0 & 0 & \frac{k_8}{2} \\ -\frac{k_3}{2} & \frac{k_3}{2} & 0 & 0 & C_2 & -\frac{k_3}{2}+\frac{k_7}{2} & -\frac{k_7}{2} & -\frac{k_7}{2} & -k_{10} \\ \frac{k_3}{2} & -\frac{k_3}{2} & 0 & -k_6 & -\frac{k_3}{2}+\frac{k_7}{2} & \frac{k_3}{2}+k_6+\frac{k_7}{2} & -\frac{k_7}{2} & -\frac{k_7}{2} & 0 \\ 0 & 0 & -k_9 & 0 & -\frac{k_7}{2} & -\frac{k_7}{2} & \frac{k_7}{2}+k_9 & \frac{k_7}{2} & 0 \\ 0 & 0 & 0 & 0 & -\frac{k_7}{2} & -\frac{k_7}{2} & \frac{k_7}{2} & \frac{k_7}{2}+k_{11} & 0 \\ 0 & 0 & -\frac{k_8}{2} & \frac{k_8}{2} & -k_{10} & 0 & 0 & 0 & \frac{k_8}{2}+k_{10} \end{bmatrix}$$

Where $C_1 = k_4 + \frac{k_5}{2} + \frac{k_8}{2} + k_9$ and $C_2 = k_2 + \frac{k_3}{2} + \frac{k_7}{2} + k_{10}$

The problem is now formulated as: $\left\{\begin{matrix} 0 \\ -60N \\ 0 \\ -110N \\ 0 \\ 0 \\ 0 \\ -60N \\ 0 \end{matrix}\right\} = K_{\{}bc\} \left\{\begin{matrix} u_{2x} \\ u_{2y} \\ u_{3x} \\ u_{3y} \\ u_{4x} \\ u_{4y} \\ u_{5x} \\ u_{5y} \\ u_{6x} \end{matrix}\right\}$

4 Solving

Now that boundary conditions have been applied, the displacements can be quickly found via gaussian elimination or by calculating the inverse of K.

4.1 Replacing variables with known values:

$$K_{bc} = 1000\frac{N}{m}\begin{bmatrix} 13.540 & -3.540 & -10 & 0 & -3.540 & 3.540 & 0 & 0 & 0 \\ -3.540 & 13.540 & 0 & 0 & 3.540 & -3.540 & 0 & 0 & 0 \\ -10 & 0 & 27.080 & 0 & 0 & 0 & -10 & 0 & -3.540 \\ 0 & 0 & 0 & 17.080 & 0 & -10 & 0 & 0 & 3.540 \\ -3.540 & 3.540 & 0 & 0 & 27.080 & 0 & -3.540 & -3.540 & -10 \\ 3.540 & -3.540 & 0 & -10 & 0 & 17.080 & -3.540 & -3.540 & 0 \\ 0 & 0 & -10 & 0 & -3.540 & -3.540 & 13.540 & 3.540 & 0 \\ 0 & 0 & 0 & 0 & -3.540 & -3.540 & 3.540 & 13.540 & 0 \\ 0 & 0 & -3.540 & 3.540 & -10 & 0 & 0 & 0 & 13.540 \end{bmatrix}$$

The displacements will be calculated via gaussian elimination (aka row-reduced echelon form):

$$\left[\begin{array}{ccccccccc|c} 13,540 & -3,540 & -10,000 & 0 & -3,540 & 3,540 & 0 & 0 & 0 & 0 \\ -3,540 & 13,540 & 0 & 0 & 3,540 & -3,540 & 0 & 0 & 0 & -60 \\ -10,000 & 0 & 27,080 & 0 & 0 & 0 & -10,000 & 0 & -3,540 & 0 \\ 0 & 0 & 0 & 17,080 & 0 & -10,000 & 0 & 0 & 3,540 & -110 \\ -3,540 & 3,540 & 0 & 0 & 27,080 & 0 & -3,540 & -3,540 & -10,000 & 0 \\ 3,540 & -3,540 & 0 & -10,000 & 0 & 17,080 & -3,540 & -3,540 & 0 & 0 \\ 0 & 0 & -10,000 & 0 & -3,540 & -3,540 & 13,540 & 3,540 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3,540 & -3,540 & 3,540 & 13,540 & 0 & -60 \\ 0 & 0 & -3,540 & 3,540 & -10,000 & 0 & 0 & 0 & 13,540 & 0 \end{array}\right]$$

4.2 Displacements

The displacements calculated are: $\begin{Bmatrix} u_{2x} \\ u_{2y} \\ u_{3x} \\ u_{3y} \\ u_{4x} \\ u_{4y} \\ u_{5x} \\ u_{5y} \\ u_{6x} \end{Bmatrix} = \begin{Bmatrix} 5.5 \\ -7.4 \\ 4.1 \\ -15.6 \\ 4.1 \\ -12.8 \\ 2.7 \\ -7.4 \\ 8.2 \end{Bmatrix} mm$

4.3 Reactions

We already have a formula for the reactive forces

$$\begin{Bmatrix} R_{1x} \\ R_{1y} \\ 0 \\ -60N \\ 0 \\ -110N \\ 0 \\ 0 \\ 0 \\ -60N \\ 0 \\ R_{6y} \end{Bmatrix} = K \begin{Bmatrix} 5.5 \\ -7.4 \\ 4.1 \\ -15.6 \\ 4.1 \\ -12.8 \\ 2.7 \\ -7.4 \\ 8.2 \end{Bmatrix} \Rightarrow \begin{Bmatrix} 0 \\ 115 \\ 0 \\ -60 \\ 0 \\ -110 \\ 0 \\ 0 \\ 0 \\ -60 \\ 0 \\ 115 \end{Bmatrix} N$$

4.4 Solving for element stress

Since stress is equal to strain times the elastic modulus:

$$\sigma = \varepsilon E$$

Since strain is the net change in an elements length, the stress can be calculated once the drift due to other nodes is subtracted, in other words[1]:

$\sigma = \frac{E}{l}\left(u_{\{}2x\} - u_{\{}1x\}\right)$ *In local coordinates*

This is easily solved using the transformation matrix that has already derived.

$$\sigma = \frac{E}{l}\begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix}\begin{Bmatrix} u_{1x} \\ u_{1y} \\ u_{2x} \\ u_{2y} \end{Bmatrix} local$$

$$\sigma = \frac{E}{l}\begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix} T_s \begin{Bmatrix} u_{1x} \\ u_{1y} \\ u_{2x} \\ u_{2y} \end{Bmatrix} global$$

The solution for each element is as follows:

4.4.1 Element 1

$$\frac{100{*}10^6 Pa}{4m}\begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0.0055 \\ -0.0074 \end{Bmatrix} = -185.0kPa$$

The negative stress indicates compressive stress.

4.4.2 Element 2

$$\frac{100{*}10^6 Pa}{4m}\begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0.0041 \\ -0.0128 \end{Bmatrix} = 102.5kPa$$

The positive stress indicates tensile stress.

4.4.3 Element 3

$$\{\frac{100{*}10^6 Pa}{5.65m}\begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{Bmatrix} 0.0055 \\ -0.0074 \\ 0.0041 \\ -0.0128 \end{Bmatrix} = 50.06kPa\}$$

4.4.4 Element 4

$$\frac{100*10^6 Pa}{4m}\begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} 0.0055 \\ -0.0074 \\ 0.0041 \\ -0.0156 \end{Bmatrix} = -35.00kPa$$

4.4.5 Element 5

$$\{\frac{100*10^6 Pa}{5.65m}\begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0.0041 \\ -0.0156 \end{Bmatrix} = -143.9kPa\}$$

4.4.6 Element 6

$$\frac{100*10^6 Pa}{4m}\begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}\begin{Bmatrix} 0.0041 \\ -0.0128 \\ 0.0041 \\ -0.0156 \end{Bmatrix} = -70.00kPa$$

4.4.7 Element 7

$$\{\frac{100*10^6 Pa}{5.65m}\begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{Bmatrix} 0.0041 \\ -0.0128 \\ 0.0027 \\ -0.0074 \end{Bmatrix} = 50.06kPa\}$$

4.4.8 Element 8

$$\{\frac{100*10^6 Pa}{5.65m}\begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{Bmatrix} 0.0041 \\ -0.0156 \\ 0.0082 \\ 0 \end{Bmatrix} = -143.9kPa\}$$

4.4.9 Element 9

$$\frac{100*10^6 Pa}{4m}\begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} 0.0041 \\ -0.0156 \\ 0.0027 \\ -0.0074 \end{Bmatrix} = -35.00kPa$$

4.4.10 Element 10

$$\frac{100*10^6 Pa}{4m}\begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} 0.0041 \\ -0.0128 \\ 0.0082 \\ 0 \end{Bmatrix} = 102.5kPa$$

4.4.11 Element 11

$$\frac{100*10^6 Pa}{4m} \begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{Bmatrix} 0.0082 \\ 0 \\ 0.0027 \\ -0.0074 \end{Bmatrix} = -185.0kPa$$

5 Conclusion and Discussion

The appendix can be referenced for the MATLAB script as written.‘

Table 1 lists the hand calculated results of the investigation/project.

Node	Dir.	Disp.(mm)	Force (N)	Element	Stress (kPa)
1	x	0	0	1	-185.0
	y	0	115 (Rcn.)	2	102.5
2	x	5.5	0	3	50.06
	y	-7.4	-60	4	-35.00
3	x	4.1	0	5	-143.9
	y	-15.6	-110	6	-70.00
4	x	4.1	0	7	50.06
	y	-12.8	0	8	-143.9
5	x	2.7	0	9	-35.00
	y	-7.4	-60	10	102.5
6	x	8.2	0	11	-185.0
	y	0	115 (Rcn.)		

Table 2 lists the MATLAB calculated results:

Node	Dir.	Disp.(mm)	Force(N)	Element	Stress(kPa)
1	x	0	0	1	-185.0
	y	0	115 (Rcn.)	2	102.0
2	x	5.5	0	3	49.92
	y	-7.4	-60	4	-35.30
3	x	4.1	0	5	-144.5
	y	-15.6	-110	6	-70.60
4	x	4.1	0	7	49.92
	y	-12.8	0	8	-144.5
5	x	2.7	0	9	-35.30
	y	-7.4	-60	10	102.2
6	x	8.2	0	11	-185.3
	y	0	115 (Rcn.)		

A careful review of both tables will show that the force and displacement values are identical. The stresses calculated by hand are slightly different than those calculated by MATLAB. This is not surprising given that the hand-calculated values were likely to propogate approximation errors at every calculation step. For this reason the MATLAB calculated results should be assumed to be the accurate results.

There’s also the fact that the hand calculated results have decimal places that are fixed; without doing the math on the displacements again it is not possible to calculate more decimals. The MATLAB results can have more decimal places reported as desired.

The engineering applications of these results are significant for several reasons:

- Truss systems are a staple of simple bridge design and have applications in high-rise construction
- The MATLAB code as written is primarily symbolic and can be adapted to any desired inputs
- A simple modification would likely allow for it to be treated as a subroutine or function
- This serves as a reminder of the importance of:
  - Establishing the necessary significant figures needed
  - Paying attention to the effect of approximation errors ahead of time. Especially where hand-calculations are concerned but also with computer calculations
- The FEM method is easily programmed symbolically, free programming modules such as SYMPY may allow for free FEM calculations where paid FEA software is cost-prohibitive

5.1 In summary:

Hand calculations have confirmed the accuracy of a programmatic implementation of FEM. This introduces the possibility of using simple programming languages (especially symbolic programming) for preliminary structural design.

6 Bibliography:

References

[1] B. Greenlee, "ME 360L - Mechanical Engineering Design III: Finite Element Trusses," University of New Mexico, ME 323, Accessed: Mar. 01, 2021. [Online]. Available: https://www.unm.edu/ bgreen/ME360/Finite

7 Appendix

[1]:
```
clear;

%% Creating Symbolic Variables for Each Truss Variable
angle = sym('t', [1,11]);
U = sym('u',[1, 12]);
F = sym('f',[1, 12]);
k_symbols = sym('k',[1,11]);
```

[2]:
```
%% Defining Scalars
A = 4e-4;
E = 100e6;%mpa

L = [4,4,5.65,4,5.65,4,5.65,5.65,4,4,4];

k_scalar = (A*E)./L;

%% Constructing numeric vectors for each variable

angle_num = [pi/2,0,-pi/4,0,pi/4,pi/2,pi/4,-pi/4,0,0,pi/2];
F_num_bc = [0,-60,0,-110,0,0,0,-60,0];

% Node in list order corresponding to each element
element_to_node=[1:4;...%1
                1,2,7,8;...%2
                3:4,7:8;...%3
                3:6;...%4
                1,2,5,6;...%5
                7:8,5:6;...%6
                7:10;...%7
                5:6,11:12;...%8
                5:6,9:10;...%9
                7:8,11:12;...%10
                11:12,9:10];%11
```

[3]:
```
%% Creating empty element matrices, for convenience
E1 = sym(zeros(12));
E2 = sym(zeros(12));
E3 = sym(zeros(12));
E4 = sym(zeros(12));
E5 = sym(zeros(12));
E6 = sym(zeros(12));
E7 = sym(zeros(12));
E8 = sym(zeros(12));
E9 = sym(zeros(12));
E10 = sym(zeros(12));
E11 = sym(zeros(12));
```

[4]:

```
%% Creating a symbolic transformation Matrix
syms T(theta) Trans(theta)
T(theta)=[cos(theta), sin(theta),0,0;...
          -sin(theta),cos(theta),0,0;...
          0,0,cos(theta),sin(theta);...
          0,0,-sin(theta),cos(theta)];
Trans(theta)=T(theta).'*[1,0,-1,0;0,0,0,0;-1,0,1,0;0,0,0,0]*T(theta);
```

[5]:

```
%% Defining stiffness matrices for each element
% This could be done with a for loop, however we would lose
% the individual element matrices
E1([1:4],[1:4])=k_symbols(1)*Trans(angle(1));
E2([1:2,7:8],[1:2,7:8])=k_symbols(2)*Trans(angle(2));
E3([3:4,7:8],[3:4,7:8])=k_symbols(3)*Trans(angle(3));
E4([3:6],[3:6])=k_symbols(4)*Trans(angle(4));
E5([1:2,5:6],[1:2,5:6])=k_symbols(5)*Trans(angle(5));
E6([7:8,5:6],[7:8,5:6])=k_symbols(6)*Trans(angle(6));
E7([7:8,9:10],[7:8,9:10])=k_symbols(7)*Trans(angle(7));
E8([5:6,11:12],[5:6,11:12])=k_symbols(8)*Trans(angle(8));
E9([5:6,9:10],[5:6,9:10])=k_symbols(9)*Trans(angle(9));
E10([7:8,11:12],[7:8,11:12])=k_symbols(10)*Trans(angle(10));
E11([11:12,9:10],[11:12,9:10])=k_symbols(11)*Trans(angle(11));
```

[6]:

```
%% Creating a global matrix
K_sym = E1+E2+E3+E4+E5+E6+E7+E8+E9+E10+E11;
```

[7]:

```
%% Converting the symbolic solution to numeric
K = subs(K_sym, [angle, k_symbols], [angle_num,k_scalar]);
```

[8]:

```
%% Boundary Condition Matrix
K_bc=K([3:11],[3:11]);
```

[9]:

```
K_num_inv = inv(K_bc);
```

[10]:
```
U_num = double(K_num_inv*F_num_bc.');%'
```

[11]:
```
U = [0,0,U_num.',0].';%'
```

[12]:
```
K_num_glob = double(K);
```

[13]:
```
F = K_num_glob*U;
```

[23]:
```
sigma = [];
difference = [-1,0,1,0];

for i = 1:11
    local_disp = [1,0,-1,0];
    nodes = element_to_node(i,:);
    EL = E/L(i);
    displacements = U(nodes);
    theta_val = angle_num(i);
    transform = T(theta_val);
    sigma(i)= EL*difference*transform*displacements;
end
sigma = sigma.';%'
```

[27]:
```
U
F
sigma
```

```
U =

         0
         0
    0.0055
   -0.0074
    0.0041
   -0.0156
```

```
    0.0041
   -0.0128
    0.0027
   -0.0074
    0.0082
         0


F =

    0.0000
  115.0000
    0.0000
  -60.0000
    0.0000
 -110.0000
         0
   -0.0000
         0
  -60.0000
   -0.0000
  115.0000


sigma =

   1.0e+05 *

   -1.8530
    1.0220
    0.4992
   -0.3530
   -1.4453
   -0.7060
    0.4992
   -1.4453
   -0.3530
    1.0220
   -1.8530
```